Hyperbolic van der Waerden and Valiant-Schrijver Conjectures

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Let $p(x_1, ..., x_n) = p(X), X \in R^n$ be a homogeneous polynomial of degree n in n real variables , $e = (1, 1, .., 1) \in R^n$ be a vector of all ones . Such polynomial p is called e-hyperbolic if for all real vectors $X \in R^n$ the univariate polynomial equation $P(te - X) = 0$ has all real roots $\lambda_1(X) \geq ... \geq \lambda_n(X)$. The number of nonzero roots $|\{i : \lambda_i(X) \neq 0\}|$ is called $Rank_p(X)$. A e-hyperbolic polynomial p is called P-hyperbolic if roots of vectors $X \in R^n_+$ with nonnegative coordinates are also nonnegative (the orthant R^n_+ belongs to the hyperbolic cone) and $p(e) > 0$. Below $\{e_1, ..., e_n\}$ stands for the canonical orthogonal basis in R^n

Theorem 1 *1. Let $p(x_1, x_2, ..., x_n)$ be a P-hyperbolic (homogeneous) polynomial of degree n ; $Rank_p(e_i) = R_i$. Define $G_i = \min(R_i, n + 1 - i)$ and assume that*

$$p(x_1, x_2, ..., x_n) \geq \prod_{1 \leq i \leq n} x_i; x_i > 0, 1 \leq i \leq n.$$

Then the following inequality holds

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \geq \prod_{1 \leq i \leq n} (\frac{G_i - 1}{G_i})^{G_i - 1} \quad (1)$$

2. If $Rank_p(e_i) \leq k \leq n$ then

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \geq (\frac{k-1}{k})^{(k-1)(n-k)} \frac{k!}{k^k} \quad (2)$$

This theorem is a vast generalization of as the van der Waerden conjecture on the permanents of doubly stochastic matrices as well of Schrijver-Valiant conjecture on the number of perfect matchings in k-regular bipartite graphs . These two famous powerful results correspond to the polynomials being products of linear forms.

Our proof is relatively simple and "noncomputational" ; it actually slightly improves Schrijver's lower bound , and uses very basic (more or less centered around Rolle's theorem) properties of hyperbolic polynomials .

The paper with the proof is available at http://lanl.arxiv.org/abs/math.CO/0504397 , see also the paper at
http://xxx.lanl.gov/abs/math.CO/0404474 .

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